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EXPLICIT CRACK-TIP FIELDS OF AN EXTENDING INTERFACE CRACK IN AN ANISOTROPIC BIMATERIAL

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Abstract—Explicit expressions for crack-tip fields of a crack, dynamically extending along the interface of an anisotropic bimaterial, are presented. The deformation considered is a combination of plane strain and anti-plane strain deformation. The amplitudes of the crack-tip fields are characterized by the stress intensity factors recently introduced by Wu (1989a, *J. Appl. Mech.*, in press). The stress intensity factors due to dislocations and body forces for a steadily moving semi-infinite crack are derived and the weight functions for crack face loading are given.

1. INTRODUCTION

The work presented here is an extension of Wu (1989a). In that work, explicit real-form expressions for σ_{i2}, $i = 1, 2, 3$, ahead of a stationary interface crack in an anisotropic bimaterial, as well as relative crack face displacements are given. These expressions contain three newly defined real-valued stress intensity factors. The stress intensity factors are defined so that as the material is homogeneous, the usual definition of the stress intensity factors is recovered. The strain energy release rate due to crack extension is given by a quadratic form of the new stress intensity factors.

In this paper, an extending interface crack in a general anisotropic bimaterial is considered. Willis (1971) investigated the energy release rate of a steadily extending interface crack using an approach different from the one employed here. He gave an expression for the energy release rate but did not derive the crack-tip fields. The simpler case of a dynamically propagating crack in a homogeneous medium was treated by Wu (1989b). Brock and Achenbach (1973) analyzed the extension of an interface crack under the influence of transient horizontally polarized shear waves. In this paper, explicit expressions for the *full* dynamic interface crack-tip fields and energy release rate are derived using the Stroh formalism for anisotropic elasticity. It is shown that similar to the static crack-tip fields considered by Wu (1989a), real-valued stress intensity factors can be defined to characterize the dynamic crack-tip fields and that the dynamic energy release rate is expressed in terms of these stress intensity factors. For comparison purposes, the expressions for the crack-tip fields and energy release rate are specialized for isotropic bimaterials for dynamic as well as static cracks.

It is shown by Wu (1988) that with the knowledge of crack-tip fields and energy release rates, the stress intensity factors due to dislocations and body forces for a steadily extending semi-infinite crack can be determined. The result is extended here to the case of a semi-infinite interface crack.

The plan of the paper is as follows. In Section 2, basic equations and the Stroh formalism are outlined. In Section 3, explicit expressions for the crack-tip fields and energy release rate for general anisotropic bimaterials are derived. In Section 4, the expressions are specialized for the case of isotropic bimaterials. Finally, stress intensity factors due to dislocations and body forces are given.

Throughout this paper, all indices range from 1 to 3 and the convention of summation over a repeated Latin index is adopted unless otherwise noted. Summation over Greek indices is expressed explicitly. Bold-faced symbols are used to represent matrices. The quantities associated with the lower half-space are distinguished by a prime.

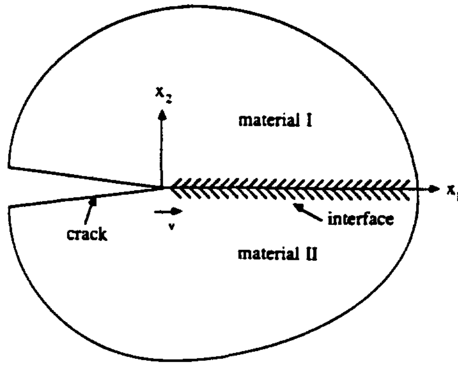

Fig. 1. An extending interface crack in anisotropic bimaterial.

2. BASIC EQUATIONS AND STROH FORMALISM

Consider a crack running along the interface in an anisotropic bimaterial. Let a moving coordinate system be attached to the crack tip. The plane of the interface is taken to be the x_1–x_3 plane and the crack front is assumed to extend infinitely in the x_3-direction. The configuration is shown in Fig. 1. It is shown by Wu (1989b) that with respect to the moving coordinate system, the equations of motion for the crack-tip fields are given by

$$C_{ijkl}\frac{\partial^2 u_k}{\partial x_l \partial x_j} = \rho v^2 \frac{\partial^2 u_i}{\partial x_1^2}, \tag{1}$$

for the upper half-space. In eqn (1), **C** is the elasticity tensor, **u** is the displacement vector, ρ is the density and v is the instantaneous crack velocity.

For subsonic motion, the general expression of the displacement vector **u** can be written as (Stroh, 1958)

$$u_k = 2\mathscr{R}\left[\sum_{\alpha=1}^{3} A_{k\alpha} f_\alpha(z_\alpha)\right], \tag{2}$$

where $\mathscr{R}$ denotes the real part, $\mathbf{A} = [\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3]$ and $z_\alpha = x_1 + p_\alpha x_2$, the vector $\mathbf{A}_\alpha$ and scalar p_α being the eigenvector and the eigenvalue with positive imaginary part, respectively, of the following equation:

$$[C_{i1k1} - \rho v^2 \delta_{ik} + (C_{i1k2} + C_{i2k1})p + C_{i2k2}p^2]A_k = 0, \tag{3}$$

where δ_{ik} is the Kronecker delta. The stresses can be expressed in terms of the stress function $\mathbf{\Phi}$ as

$$\sigma_{i1} = -\frac{\partial \Phi_i}{\partial x_2} + \rho v^2 \frac{\partial u_i}{\partial x_1}, \tag{4}$$

$$\sigma_{i2} = \frac{\partial \Phi_i}{\partial x_1}. \tag{5}$$

The stress function $\mathbf{\Phi}$ can be represented as

$$\Phi_k = 2\mathscr{R}\left[\sum_{\alpha=1}^{3} B_{k\alpha} f_\alpha(z_\alpha)\right], \tag{6}$$

where the matrix **B** is given by

$$B_{i\alpha} = (C_{k1i2}+p_\alpha C_{i2k2})A_{k\alpha}. \tag{7}$$

The expressions for $\mathbf{u}'$ and $\mathbf{\Phi}'$ for the lower half-space can be obtained by adding primes to the quantities in eqns (2) and (6).

3. CRACK-TIP FIELDS

In this section the crack-tip fields of an extending interface crack are derived using the Stroh formalism discussed in the last section.

The functions $f_\alpha(z_\alpha)$ and $f'_\alpha(z'_\alpha)$ for which the tractions are continuous across $x_2 = 0$ and the displacements are continuous across the interface $x_1 > 0$ and $x_2 = 0$, can be expressed as

$$f_\alpha(z_\alpha) = B^{-1}_{\alpha j} N_{jk} \Psi_k(z_\alpha), \tag{8}$$

$$f'_\alpha(z'_\alpha) = -B'^{-1}_{\alpha j} \bar{N}_{jk} \Psi_k(z'_\alpha), \tag{9}$$

where the function $\mathbf{\Psi}(z)$ has the following property

$$\mathbf{\Psi}(z) = -\overline{\mathbf{\Psi}(\bar{z})}. \tag{10}$$

In eqns (8) and (9), the matrix $\mathbf{N}$ is given by

$$\mathbf{N} = \mathbf{M}^{-1}, \tag{11}$$

$$\mathbf{M} = \mathbf{A}\mathbf{B}^{-1} - \bar{\mathbf{A}}'\bar{\mathbf{B}}'^{-1}. \tag{12}$$

The matrix $\mathbf{M}$ can be expressed as (Ting, 1986)

$$\mathbf{M} = -(\mathbf{W}+i\mathbf{D}), \tag{13}$$

where $\mathbf{W}$ is antisymmetric and $\mathbf{D}$ is symmetric. Both $\mathbf{W}$ and $\mathbf{D}$ are real matrices. Similarly, the matrix $\mathbf{N}$ can be expressed as

$$\mathbf{N} = \tilde{\mathbf{W}}+i\tilde{\mathbf{D}}, \tag{14}$$

where $\tilde{\mathbf{W}}$ is antisymmetric and $\tilde{\mathbf{D}}$ is symmetric.

Substituting eqn (8) into eqn (6) and imposing traction-free conditions for the upper crack face leads to

$$\mathbf{N}\mathbf{\Psi}^+(x_1) - \bar{\mathbf{N}}\mathbf{\Psi}^-(x_1) = \mathbf{0}. \tag{15}$$

For bounded displacements, the functions Ψ_k satisfying eqn (15) and yielding singular stress field can be expressed as

$$\mathbf{\Psi}(z) = -\frac{i}{\sqrt{2\pi}}\tilde{\mathbf{D}}^{-1}(\bar{\mathbf{Q}}^T)^{-1}\mathbf{\Lambda}(z)\bar{\mathbf{Q}}^T\mathbf{K}, \tag{16}$$

where

$$\mathbf{\Lambda}(z) = \sqrt{z}\,\mathrm{diag}\left[\frac{1}{1+2i\gamma}\left(\frac{z}{\hat{r}}\right)^{i\gamma}, \frac{1}{1-2i\gamma}\left(\frac{z}{\hat{r}}\right)^{-i\gamma}, 1\right], \tag{17}$$

and $\mathbf{K}$ is a real-valued constant vector defined as the interface stress intensity factor. In eqn (17), $\hat{r}$ is an arbitrary length scale and $\mathbf{Q} = [\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3]$ where $\mathbf{q}_k$ is the eigenvector of the following equation:

$$(\tilde{\mathbf{W}}+i\lambda\tilde{\mathbf{D}})\mathbf{q} = 0. \tag{18}$$

The eigenvalues of λ of eqn (18) satisfy (Ting, 1986)

$$\lambda^3-\beta^2\lambda = 0, \tag{19}$$

where β is given by

$$\beta = \{-\tfrac{1}{2}\mathrm{tr}[(\mathbf{WD}^{-1})^2]\}^{1/2}. \tag{20}$$

The bimaterial constant γ is related to β by

$$\gamma = \frac{1}{2\pi}\log\frac{1+\beta}{1-\beta}. \tag{21}$$

The matrix $\mathbf{Q}$ is chosen such that (Wu, 1989a):

$$\bar{\mathbf{Q}}^T\tilde{\mathbf{D}}\mathbf{Q} = \mathbf{I}, \tag{22}$$

$$\bar{\mathbf{Q}}^T\tilde{\mathbf{W}}\mathbf{Q} = i\beta\,\mathrm{diag}\,[1,-1,0], \tag{23}$$

where diag stands for diagonal matrix. From eqns (22) and (23) one can show that

$$(\bar{\mathbf{Q}}^T)^{-1}\mathrm{diag}\,[1,-1,0]\bar{\mathbf{Q}}^T = -i\mathbf{P}, \tag{24}$$

where

$$\mathbf{P} = \frac{1}{\beta}\tilde{\mathbf{W}}\tilde{\mathbf{D}}^{-1}. \tag{25}$$

Using eqn (25), Wu (1989a) showed that

$$\begin{aligned}\mathbf{R}[c] &= (\bar{\mathbf{Q}}^T)^{-1}\mathrm{diag}\,[c,\bar{c},1](\bar{\mathbf{Q}}^T),\\ &= \mathbf{I}+\mathscr{I}[c]\mathbf{P}+(1-\mathscr{R}[c])\mathbf{P}^2,\end{aligned} \tag{26}$$

where $\mathscr{I}$ denotes the imaginary part. Using eqn (26), Wu (1989a) obtained the real-form expressions for σ_{i2} ahead of the crack (i.e. $x_2 = 0$, $x_1 > 0$) and the relative crack face displacements.

It is shown here that eqn (26) can be generalized for any diagonal matrix $\Lambda = [\lambda_1,\lambda_2,\lambda_3]$. Let Λ be rewritten as

$$\Lambda = \lambda_3\mathbf{I}+\frac{\lambda_1-\lambda_2}{2}\mathrm{diag}\,[1,-1,0]+\left(\frac{\lambda_1+\lambda_2}{2}-\lambda_3\right)\mathrm{diag}\,[1,1,0]. \tag{27}$$

Using eqn (24), one then has

$$(\bar{\mathbf{Q}}^T)^{-1}\Lambda(\bar{\mathbf{Q}}^T) = \lambda_3\mathbf{I}-i\frac{\lambda_1-\lambda_2}{2}\mathbf{P}+\left(\lambda_3-\frac{\lambda_1+\lambda_2}{2}\right)\mathbf{P}^2. \tag{28}$$

From the Hayley–Hamilton theorem that a matrix satisfies its own characteristic equation, one can show, from eqns (18) and (19), that

$$\mathbf{P}^3+\mathbf{P} = \mathbf{0}. \tag{29}$$

From eqns (14) and (25), one also has

$$\mathbf{N}\tilde{\mathbf{D}}^{-1} = i\mathbf{I}+\beta\mathbf{P}. \tag{30}$$

With eqns (28), (29) and (30), one can show that

$$\mathbf{N}\mathbf{\Psi}(z_\alpha) = \frac{1}{\sqrt{2\pi}}[g_0(z_\alpha)\mathbf{I}+g_1(z_\alpha)\mathbf{P}+g_2(z_\alpha)\mathbf{P}^2], \tag{31}$$

where

$$g_0(z_\alpha) = \sqrt{z_\alpha}, \tag{32}$$

$$g_1(z_\alpha) = \frac{\sqrt{z_\alpha}}{i2\cosh(\pi\gamma)}\left(\frac{\omega}{1+2i\gamma}\left(\frac{z_\alpha}{\hat{r}}\right)^{i\gamma}-\frac{1}{\omega(1-2i\gamma)}\left(\frac{z_\alpha}{\hat{r}}\right)^{-i\gamma}\right), \tag{33}$$

$$g_2(z_\alpha) = \sqrt{z_\alpha}-\frac{\sqrt{z_\alpha}}{2\cosh(\pi\gamma)}\left(\frac{\omega}{1+2i\gamma}\left(\frac{z_\alpha}{\hat{r}}\right)^{i\gamma}+\frac{1}{\omega(1-2i\gamma)}\left(\frac{z_\alpha}{\hat{r}}\right)^{-i\gamma}\right) \tag{34}$$

and

$$\omega = e^{\pi\gamma}. \tag{35}$$

Substituting eqn (31) into eqn (8), and eqn (8) into eqn (6) leads to

$$\mathbf{\Phi} = \sqrt{\frac{2}{\pi}}\mathscr{R}\left[\sum_{n=0}^{2}\mathbf{B}\mathbf{\Lambda}_n(\mathbf{z})\mathbf{B}^{-1}\mathbf{P}^n\right]\mathbf{K}, \tag{36}$$

where

$$\mathbf{\Lambda}_n = \text{diag}[g_n(z_1), g_n(z_2), g_n(z_3)]. \tag{37}$$

Similarly, the displacement vector is given by

$$\mathbf{u} = \sqrt{\frac{2}{\pi}}\mathscr{R}\left[\sum_{n=0}^{2}\mathbf{A}\mathbf{\Lambda}_n(\mathbf{z})\mathbf{B}^{-1}\mathbf{P}^n\right]\mathbf{K}. \tag{38}$$

Following a similar procedure, one can show that the stress function and displacement vector for the lower material are also given by eqns (36) and (38), respectively, if ω is replaced by ω' defined as

$$\omega' = \frac{1}{\omega} = e^{-\pi\gamma}, \tag{39}$$

and **A**, **B** and z_α are replaced by **A**′, **B**′ and z'_α, respectively.

Note that since the functions g_1 and g_2 in eqns (36) and (38) contain an arbitrary length parameter $\hat{r}$, the stress intensity factor **K** also depends on $\hat{r}$. In fact, it can be shown that

$$\mathbf{K}^{(1)} = \mathbf{R}\left[\left(\frac{\hat{r}_1}{\hat{r}_2}\right)^{i\gamma}\right]\mathbf{K}^{(2)}, \tag{40}$$

where $\mathbf{K}^{(1)}$ and $\mathbf{K}^{(2)}$ are the stress intensity factors corresponding to $\hat{r} = \hat{r}_1$ and $\hat{r} = \hat{r}_2$, respectively.

Equations (36) and (38) are the expressions of the complete crack-tip fields for the extending interface crack. If the medium is homogeneous, i.e. $\gamma = 0$, the crack-tip field reduces to that derived by Wu (1989b). Along the interface, from eqns (36) and (5), the stresses $t_i = \sigma_{i2}$ are given by

$$\mathbf{t} = \frac{1}{\sqrt{2\pi}}\mathscr{R}\left[\sum_{n=0}^{2}\left[\frac{\partial g_n(x_1)}{\partial x_1}\mathbf{P}^n\right]\mathbf{K}\right],$$

$$= \frac{1}{\sqrt{2\pi r}}\mathbf{R}\left[\left(\frac{r}{\hat{r}}\right)^{i\gamma}\right]\mathbf{K}, \tag{41}$$

thus agreeing with the result of (Wu, 1989a). In eqn (41), the matrix function **R** is given by eqn (26) and r is the distance from the crack-tip. Setting $r = \hat{r}$ in eqn (41), one has

$$\mathbf{t} = \frac{1}{\sqrt{2\pi\hat{r}}}\mathbf{K}. \tag{42}$$

Equation (42) can be regarded as an alternative definition of **K**. The crack face displacement vector for the upper crack face is simplified to

$$\mathbf{u} = \sqrt{\frac{2r}{\pi}}\mathbf{L}^{-1}\mathbf{R}\left[\frac{1}{\cosh(\pi\gamma)(1+2i\gamma)}\left(\frac{r}{\hat{r}}\right)^{i\gamma}\right]\mathbf{K}, \tag{43}$$

where

$$\mathbf{L}^{-1} = -\mathscr{I}[\mathbf{A}\mathbf{B}^{-1}]. \tag{44}$$

The crack face displacement vector for the lower crack face is

$$\mathbf{u}' = -\sqrt{\frac{2r}{\pi}}\mathbf{L}'^{-1}\mathbf{R}\left[\frac{1}{\cosh(\pi\gamma)(1+2i\gamma)}\left(\frac{r}{\hat{r}}\right)^{i\gamma}\right]\mathbf{K}. \tag{45}$$

It should be pointed out that in the foregoing analysis, it was tacitly assumed that the matrices **B**, **B**′ and **M** can be inverted. This is true as long as the crack velocity v is lower than the minimum of the critical velocities V_R, V'_R and V_{ST} for which $\det[\mathbf{B}(V_R)] = 0$, $\det[\mathbf{B}'(V'_R)] = 0$ and $\det[\mathbf{M}(V_{ST})] = 0$. The critical velocities V_R, V'_R are the Rayleigh wave speeds for the upper and lower material, respectively, and V_{ST} is the Stonely wave speed for the bimaterial (Barnett and Lothe, 1974). It is interesting to note that as the bimaterial constant β satisfies (Ting, 1986)

$$\det[\mathbf{W} + i\beta\mathbf{D}] = 0, \tag{46}$$

the condition for V_{ST} implies that at V_{ST}, $\beta = 1$ and the corresponding γ becomes unbounded.

The energy release rate due to crack extension can be derived by a similar procedure outlined in Wu (1989a). The derivation will not be repeated here and the result is given by

$$G_d = \tfrac{1}{4}\mathbf{K}^T\tilde{\mathbf{D}}^{-1}\mathbf{K}. \tag{47}$$

Note that although the stress intensity factors are dependent on the length scale $\hat{r}$, the energy release rate is independent of the choices of $\hat{r}$. To see this fact, let $\mathbf{K}^{(1)}$ and $\mathbf{K}^{(2)}$ be the stress intensity factors corresponding to $\hat{r} = \hat{r}_1$ and $\hat{r} = \hat{r}_2$, respectively. From eqn (40), one can show

$$(\mathbf{K}^{(1)})^T\tilde{\mathbf{D}}^{-1}\mathbf{K}^{(1)} = (\mathbf{K}^{(2)})^T\mathbf{R}^T\left[\left(\frac{\hat{r}_1}{\hat{r}_2}\right)^{i\tilde{\varepsilon}}\right]\tilde{\mathbf{D}}^{-1}\mathbf{R}\left[\left(\frac{\hat{r}_1}{\hat{r}_2}\right)^{i\tilde{\varepsilon}}\right]\mathbf{K}^{(2)},$$

$$= (\mathbf{K}^{(2)})^T\tilde{\mathbf{D}}^{-1}\mathbf{K}^{(2)}.$$

Thus eqn (47) is independent of $\hat{r}$.

4. ISOTROPIC BIMATERIALS

In this section, the crack-tip fields and associated energy release rate are specialized for isotropic bimaterials. For isotropic bimaterials, the plane deformation and anti-plane deformation can be uncoupled. Only the plane deformation is considered here. As discussed in the previous section, the expressions for the stress function and displacement vector are similar for the upper and lower half-spaces, so attention need only be paid to the upper half-space.

The matrices $\mathbf{A}$ and $\mathbf{B}$ are given by (Wu, 1989b)

$$\mathbf{A} = \begin{bmatrix} 1 & -i\beta_2 \\ i\beta_1 & 1 \end{bmatrix}, \tag{48}$$

$$\mathbf{B} = \mu\begin{bmatrix} 2i\beta_1 & 1+\beta_2^2 \\ -(1+\beta_2^2) & 2i\beta_2 \end{bmatrix}, \tag{49}$$

where

$$\beta_\alpha = \sqrt{1-\left(\frac{v}{V_\alpha}\right)^2}, \tag{50}$$

μ is the shear modulus, V_1 is the longitudinal wave velocity and V_2 is the shear wave velocity. The eigenvalue p_α is given by $i\beta_\alpha$. The inverse matrix of $\mathbf{B}$ is

$$\mathbf{B}^{-1} = -\frac{1}{\mu\Delta}\begin{bmatrix} 2i\beta_2 & -(1+\beta_2^2) \\ (1+\beta_2^2) & 2i\beta_1 \end{bmatrix}, \tag{51}$$

where $\Delta = 4\beta_1\beta_2-(1+\beta_2^2)^2$. The matrices $\mathbf{W}$ and $\mathbf{D}$ are given by

$$\mathbf{W} = \begin{bmatrix} 0 & w \\ -w & 0 \end{bmatrix}, \tag{52}$$

$$\mathbf{D} = \begin{bmatrix} d_1 & 0 \\ 0 & d_2 \end{bmatrix}, \tag{53}$$

where

$$w = \frac{1}{\mu'\Delta'}(1+\beta_2'^2-2\beta_1'\beta_2') - \frac{1}{\mu\Delta}(1+\beta_2^2-2\beta_1\beta_2), \tag{54}$$

$$d_1 = \frac{1}{\mu\Delta}\beta_2(1-\beta_2^2)+\frac{1}{\mu'\Delta'}\beta_2'(1-\beta_2'^2), \tag{55}$$

$$d_2 = \frac{1}{\mu\Delta}\beta_1(1-\beta_2^2)+\frac{1}{\mu'\Delta'}\beta_1'(1-\beta_2'^2). \tag{56}$$

From eqn (20), the bimaterial constant β is

$$\beta = \frac{w}{\sqrt{d_1 d_2}}. \tag{57}$$

The matrix $\mathbf{P}$ defined by eqn (25) is

$$\mathbf{P} = \begin{bmatrix} 0 & \sqrt{\dfrac{d_2}{d_1}} \\ -\sqrt{\dfrac{d_1}{d_2}} & 0 \end{bmatrix}, \tag{58}$$

and

$$\mathbf{P}^2 = -\mathbf{I}. \tag{59}$$

The matrix $\tilde{\mathbf{D}}^{-1}$ in eqn (47) is given by

$$\tilde{\mathbf{D}}^{-1} = (1-\beta^2)\mathbf{D}. \tag{60}$$

With eqn (59), the stress function given by eqn (36) can be simplified to

$$\mathbf{\Phi} = \sqrt{\frac{2}{\pi}}(\mathscr{R}[\mathbf{B}\tilde{\mathbf{\Lambda}}_0(z)\mathbf{B}^{-1}] + \mathscr{I}[\mathbf{B}\tilde{\mathbf{\Lambda}}_1(z)\mathbf{B}^{-1}]\mathbf{P})\mathbf{K}, \tag{61}$$

where

$$\tilde{\mathbf{\Lambda}}_\alpha = \operatorname{diag}[h_\alpha(z_1), h_\alpha(z_2)], \tag{62}$$

$$h_0(z_\alpha) = g_0(z_\alpha) - g_2(z_\alpha), \tag{63}$$

$$h_1(z_\alpha) = ig_1(z_\alpha), \tag{64}$$

and g_0, g_1, g_2 are given by eqns (32), (33) and (34), respectively. Similarly, the displacement vector given by eqn (38) becomes

$$\mathbf{u} = \sqrt{\frac{2}{\pi}}(\mathscr{R}[\mathbf{A}\tilde{\mathbf{\Lambda}}_0(z)\mathbf{B}^{-1}] + \mathscr{I}[\mathbf{A}\tilde{\mathbf{\Lambda}}_1(z)\mathbf{B}^{-1}]\mathbf{P})\mathbf{K}. \tag{65}$$

Equation (65) can also be written in the component form as

$$u_i = \sqrt{\frac{2}{\pi}}\frac{1}{\mu\Delta}\sum_{\alpha=1}^{2}\left(\mathscr{R}[U_{i\alpha}] - \sum_{\beta=1}^{2}\varepsilon_{3\alpha\beta}\sqrt{\frac{d_\alpha}{d_\beta}}\mathscr{I}[\hat{U}_{i\beta}]\right)K_\alpha, \tag{66}$$

where ε is the permutation tensor and $U_{i\alpha}$ is given by

$$U_{11} = -i\beta_2(2h_0(z_1) - (1+\beta_2^2)h_0(z_2)), \tag{67}$$

$$U_{12} = (1+\beta_2^2)h_0(z_1) - 2\beta_1\beta_2 h_0(z_2), \tag{68}$$

$$U_{21} = 2\beta_1\beta_2 h_0(z_1) - (1+\beta_2^2)h_0(z_2), \tag{69}$$

$$U_{22} = -i\beta_1(2h_0(z_2) - (1+\beta_2^2)h_0(z_1)), \tag{70}$$

and $\hat{U}_{i\alpha}$ is obtained from $U_{i\alpha}$ by replacing h_0 with h_1. From eqns (4), (5) and (61), the stresses can be expressed as

$$\sigma_{ij} = \sqrt{\frac{2}{\pi}}\frac{1}{\Delta}\sum_{\alpha=1}^{2}\left(\mathscr{R}[S_{ij\alpha}] - \sum_{\beta=1}^{2}\varepsilon_{3\alpha\beta}\sqrt{\frac{d_\alpha}{d_\beta}}\mathscr{I}[\hat{S}_{ij\beta}]\right)K_\alpha, \tag{71}$$

where $S_{ij\alpha}$ is given by

$$S_{111} = -2i\beta_1[(1+2\beta_1^2-\beta_2^2)h_0^*(z_1)-(1+\beta_2^2)h_0^*(z_2)], \tag{72}$$

$$S_{112} = (1+\beta_2^2)(1+2\beta_1^2-\beta_2^2)h_0^*(z_1)-4\beta_1\beta_2h_0^*(z_2), \tag{73}$$

$$S_{121} = 4\beta_1\beta_2h_0^*(z_1)-(1+\beta_2^2)^2h_0^*(z_2), \tag{74}$$

$$S_{122} = -i2\beta_1(1+\beta_2^2)(h_0^*(z_2)-h_0^*(z_1)), \tag{75}$$

$$S_{221} = -i2\beta_2(1+\beta_2^2)(h_0^*(z_2)-h_0^*(z_1)), \tag{76}$$

$$S_{222} = 4\beta_1\beta_2h_0^*(z_2)-(1+\beta_2^2)^2h_0^*(z_1), \tag{77}$$

the function h_0^* being

$$h_0^*(z_\alpha) = \frac{\mathrm{d}h_0(z_\alpha)}{\mathrm{d}z_\alpha}, \\ = \frac{1}{4\cosh(\pi\gamma)\sqrt{z_\alpha}}\left(\omega\left(\frac{z_\alpha}{r}\right)^{i\gamma}+\frac{1}{\omega}\left(\frac{z_\alpha}{r}\right)^{-i\gamma}\right), \tag{78}$$

and $\hat{S}_{ij\alpha}$ is obtained from $S_{ij\alpha}$ by replacing h_0^* with h_1^* given by

$$h_1^*(z_\alpha) = \frac{\mathrm{d}h_1(z_\alpha)}{\mathrm{d}z_\alpha}, \\ = \frac{1}{4\cosh(\pi\gamma)\sqrt{z_\alpha}}\left(\omega\left(\frac{z_\alpha}{r}\right)^{i\gamma}-\frac{1}{\omega}\left(\frac{z_\alpha}{r}\right)^{-i\gamma}\right). \tag{79}$$

If the upper and the lower material are identical, $h_0(z_\alpha) = \sqrt{z_\alpha}$, $h_0^*(z_\alpha) = 1/2\sqrt{z_\alpha}$, $h_1 = h_1^* = 0$, and eqns (66) and (71) reduce to the results given by Freund (1976). The crack-tip fields of a stationary interface crack can not be obtained directly from eqns (61) and (65) because $\Delta = 0$ as $v = 0$. However, if proper limits as $v \to 0$ are taken, it can be shown that

$$\lim_{v\to 0}\mathbf{B}\,\mathrm{diag}\,[f(z_1), f(z_2)]\mathbf{B}^{-1} = f(z)\mathbf{I}+x_2\frac{\mathrm{d}f(z)}{\mathrm{d}z}\begin{bmatrix} i & -1 \\ -1 & -i\end{bmatrix}, \tag{80}$$

$$\lim_{v\to 0}\mathbf{A}\,\mathrm{diag}\,[f(z_1), f(z_2)]\mathbf{B}^{-1} = \frac{1}{4\mu}\left(f(z)\begin{bmatrix} -i(\kappa+1) & \kappa-1 \\ -(\kappa-1) & -i(\kappa+1)\end{bmatrix}\right. \\ \left. +2x_2\frac{\mathrm{d}f(z)}{\mathrm{d}z}\begin{bmatrix} 1 & i \\ i & -1\end{bmatrix}\right), \tag{81}$$

where $\kappa = 3-4\nu$, ν being the Poisson ratio, and $z = x_1+ix_2$. The limiting form of $\mathbf{P}$ is given by

$$\lim_{r\to 0} \mathbf{P} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}. \tag{82}$$

The stationary crack-tip fields obtained by substituting eqns (80), (81) and (82) into eqns (61) and (65) agree with those derived by Rice and Sih (1965) and by Chen (1983) if **K** is replaced by $\sqrt{\pi}\cosh(\pi\gamma)\mathbf{K}$.

5. S.I.F. DUE TO BODY FORCES AND DISLOCATIONS

In this section the stress intensity factors due to body forces and dislocations for a semi-infinite crack are given. The body forces, dislocation and crack are assumed to translate in the x_1-direction at a constant velocity v.

The problem in which a crack propagates in an homogeneous medium has been analyzed by Wu (1988) by examining the interaction between the body forces or dislocations and the crack. The treatment of that work can be extended to the present problem and the result can be shown to be

$$K_p = -2\tilde{D}_{pq}\left(\int_A \left(\frac{\partial\hat{\Phi}_{iq}}{\partial x_1}\alpha_i + \frac{\partial\hat{u}_{iq}}{\partial x_1}f_i\right)\mathrm{d}A + \int_{A'}\left(\frac{\partial\hat{\Phi}'_{iq}}{\partial x_1}\alpha'_i + \frac{\partial\hat{u}'_{iq}}{\partial x_1}f'_i\right)\mathrm{d}A\right), \tag{83}$$

where α_i is the dislocation density and f_i is the body force density; A and A' denote the upper and lower halfspaces, respectively; the functions $\hat{\Phi}_{iq}$ and $\hat{u}_{iq}$ are the universal functions in the crack-tip fields of eqns (36) and (38) for the upper material:

$$\Phi_i = \hat{\Phi}_{iq}K_q, \tag{84}$$

$$u_i = \hat{u}_{iq}K_q, \tag{85}$$

and similarly $\hat{\Phi}'_{iq}$ and $\hat{u}'_{iq}$ are the universal functions in the crack-tip fields for the lower material.

From eqns (83) and (43), the stress intensity factors due to loading **T** on the upper crack face is obtained as

$$\mathbf{K} = \int_0^{\infty} \mathbf{H}(r)\mathbf{T}(r)\,\mathrm{d}r, \tag{86}$$

where $\mathbf{H}(r)$ is given by

$$\mathbf{H}(r) = \sqrt{\frac{2}{\pi r}}\,\tilde{\mathbf{D}}\mathbf{R}^T\left[\frac{1}{\cosh(\pi\gamma)}\left(\frac{r}{\hat{r}}\right)^{i\gamma}\right]\mathbf{L}^{-1}, \tag{87}$$

as the weight function for **T**. For homogeneous media ($\gamma = 0$), it can be shown that

$$\tilde{\mathbf{D}} = \tfrac{1}{2}\mathbf{L},$$

and eqn (87) reduces to

$$\mathbf{H}(r) = \frac{1}{\sqrt{2\pi r}}\mathbf{I}, \tag{88}$$

which is independent of the material constants. This fact has been reached by Wu (1988). From eqn (45), the weight function $\mathbf{H}'$ for loading at the lower crack face can be shown to be

$$\mathbf{H}' = -\sqrt{\frac{2}{\pi r}}\,\tilde{\mathbf{D}}\mathbf{R}^T\left[\frac{1}{\cosh(\pi\gamma)}\left(\frac{r}{\bar{r}}\right)^{i\gamma}\right]\mathbf{L}'^{-1}. \tag{89}$$

The weight function $\mathbf{H}^*$ for self-equilibrating tractions on the crack face is simply given by the sum of $\mathbf{H}$ and $-\mathbf{H}'$:

$$\mathbf{H}^* = \mathbf{H}-\mathbf{H}',$$

$$= \sqrt{\frac{2}{\pi r}}\,\tilde{\mathbf{D}}\mathbf{R}^T\left[\frac{1}{\cosh(\pi\gamma)}\left(\frac{r}{\bar{r}}\right)^{i\gamma}\right]\mathbf{D}. \tag{90}$$

Using the following identities:

$$\mathbf{R}[c]\mathbf{R}[d] = \mathbf{R}[cd], \tag{91}$$

$$\mathbf{D} = \tilde{\mathbf{D}}^{-1}\mathbf{R}[\cosh^2(\pi\gamma)], \tag{92}$$

$$\tilde{\mathbf{D}}\mathbf{P}^T\tilde{\mathbf{D}}^{-1} = -\mathbf{P}, \tag{93}$$

eqn (90) can be simplified as

$$\mathbf{H}^* = \sqrt{\frac{2}{\pi r}}\,\mathbf{R}\left[\cosh(\pi\gamma)\left(\frac{r}{\bar{r}}\right)^{-i\gamma}\right]. \tag{94}$$

Consider the loading used in Willis (1971):

$$T_j = \frac{\lambda_j^2}{(r+\lambda_j)^2}\,\bar{T}_j \quad \text{(no summation on } j\text{)}, \tag{95}$$

where λ_j is a constant length parameter and $\bar{T}_j$ a constant reference traction. For isotropic interface cracks, the stress intensity factors were calculated to be

$$K_1 = \sqrt{2\pi}\left[\left(\frac{1}{2}\cos\left(\gamma\log\left(\frac{\lambda_1}{\bar{r}}\right)\right)+\gamma\sin\left(\gamma\log\left(\frac{\lambda_1}{\bar{r}}\right)\right)\right)\sqrt{\lambda_1}\bar{T}_1 - \sqrt{\frac{d_2}{d_1}}\left(\frac{1}{2}\sin\left(\gamma\log\left(\frac{\lambda_2}{\bar{r}}\right)\right)-\gamma\cos\left(\gamma\log\left(\frac{\lambda_2}{\bar{r}}\right)\right)\right)\sqrt{\lambda_2}\bar{T}_2\right], \tag{96}$$

$$K_2 = \sqrt{2\pi}\left[\left(\frac{1}{2}\cos\left(\gamma\log\left(\frac{\lambda_2}{\bar{r}}\right)\right)+\gamma\sin\left(\gamma\log\left(\frac{\lambda_2}{\bar{r}}\right)\right)\right)\sqrt{\lambda_2}\bar{T}_2 + \sqrt{\frac{d_1}{d_2}}\left(\frac{1}{2}\sin\left(\gamma\log\left(\frac{\lambda_1}{\bar{r}}\right)\right)-\gamma\cos\left(\gamma\log\left(\frac{\lambda_1}{\bar{r}}\right)\right)\right)\sqrt{\lambda_1}\bar{T}_1\right], \tag{97}$$

where d_1 and d_2 are given by eqns (55) and (56), respectively. From eqns (47) and (60), the corresponding energy release rate is given by

$$G_d = \frac{\pi}{8}\frac{(1+4\gamma^2)}{\cosh^2(\pi\gamma)}\left(d_1\lambda_1\bar{T}_1^2+d_2\lambda_2\bar{T}_2^2+2\sin\left(\gamma\log\left(\frac{\lambda_2}{\lambda_1}\right)\right)\sqrt{d_1d_2\lambda_1\lambda_2}\,\bar{T}_1\bar{T}_2\right). \tag{98}$$

The above result is different from that obtained by Willis (1971). It is believed that Willis' result is questionable as his expression does not reduce to the correct form for homogeneous media. For the special case: $\lambda_1 = \lambda_2 = \lambda$, $\bar{T}_1 = \bar{T}_2 = \bar{T}$, eqn (98) becomes

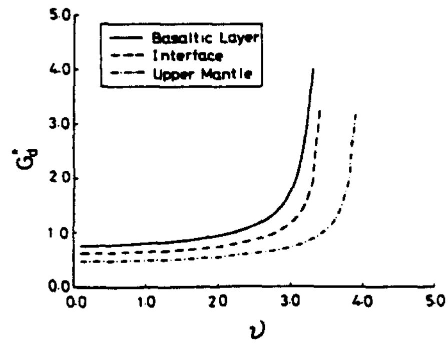

Fig. 2. G_d^* as a function of v for the Moho interface, basaltic layer and upper mantle of the earth.

$$G_d^* = G_d \frac{8}{\pi\lambda \bar{T}^2} = \frac{(1+4\gamma^2)}{\cosh^2(\pi\gamma)}(d_1+d_2). \tag{99}$$

For comparison purposes, G_d^* has been plotted in Fig. 2 as a function of v for a particular interface considered by Willis (1971) with $\beta_1 = 6.5$, $\beta_2 = 3.74$, $\beta_1' = 7.76$, $\beta_2' = 4.36$ km s^{-1}, $\rho = 2.85$ and $\rho' = 3.3$ g cm^{-3}. These values correspond to the Moho interface between the basaltic layer and the upper mantle of the earth (Jeffreys, 1970). Also plotted on Fig. 2 are the values G_d^* would have if the crack were propagating through a homogeneous material with properties of either the basaltic layer or the upper mantle. For v less than 2 km s^{-1}, it is seen from Fig. 2 that the interface G_d^* is nearly equal to the average of the G_d^* for the basaltic layer and the upper mantle. Also, for v less than 2 km s^{-1}, the values of G_d^* are almost the same as the values for the stationary crack. The inertial effects become significant as v approaches V_{ST} for the interface, V_R for the basaltic layer and V_R' for the upper mantle. Figure 2 shows that the values of V_{ST} and V_R are almost the same and are given by 3.4 km s^{-1} approximately and V_R' is about 4.0 km s^{-1}.

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